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1995 J. Phys. A: Math. Gen. 28 L639

(http://iopscience.iop.org/0305-4470/28/24/003)

J. Phys. A: Math. Gen. **28** (1995) L639–L643. Printed in the UK

LETTER TO THE EDITOR

The dynamic critical exponent of dilute and pure Ising systems

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Received 6 September 1995

Abstract. A massive dynamic field theory is employed to calculate the critical exponent z of the three-dimensional dilute Ising model to two-loop order without expansion in $\sqrt{4-d}$ (with result $z \simeq 2.191$). Applying the same method to the pure Ising model at three-loop order we find $z = 2.022$ for $d = 3$ and $z = 2.124$ for $d = 2$.

The most reliable field theoretic estimates of the dynamic critical exponent of the pure Ising model in $d = 2$ and $d = 3$ dimensions have been obtained by interpolating expansions in $\epsilon = 4 - d$ and $\epsilon' = d - 1$ to second order in ϵ and ϵ', respectively [1]. The expansion around the lower critical dimension is based on a model describing the relaxation dynamics of an interface below T_c. Since up to now there is no field theoretic model for an interface of dilute Ising systems estimates of the dynamic exponent in the presence of impurities are less reliable. Moreover, in the case of the dilute Ising model the usual ϵ-expansion is not possible [2]. An expansion in $\sqrt{\epsilon}$ to second order yields [3]

$$z = 2 + 0.336\sqrt{\epsilon}(1 - 0.932\sqrt{\epsilon}) + O(\epsilon^{3/2}) \tag{1}$$

with a relatively large $O(\epsilon)$-contribution. In [3] the $\sqrt{\epsilon}$-expansion was improved by a Padé–Borel approximation for the Callan–Symanzik functions at three-loop order to obtain the three-dimensional estimate $z \simeq 2.180$.

Prudnikov and Vakilov (PV) have calculated z directly for $d = 3$, i.e. without expansion in powers of $\sqrt{\epsilon}$ [4]. However, they applied a massless renormalization scheme which leads to infrared divergences in dimensions $d < 4$. At two-loop order the only divergence occurs at $d = 2$ but it is easy to check that the three-loop contribution (which was not considered by PV) is divergent for $d = 3$. For the pure Ising model this divergence occurs at four-loop order.

In this letter the dynamic exponent is calculated up to two-loop order in the framework of a massive field theory which is well defined for $d = 2$ and $d = 3$ at every order of the perturbation theory [5]. For the pure Ising model the calculation is extended to three-loop order and the results are compared with the exponents given in [1].

The purely relaxational dynamics of an order parameter field $s(r, t)$ in the presence of random impurities can be expressed in the form of the Langevin equation

$$\partial_t s(r,t) = -\lambda \left[(\tau + \psi(r))\, s(r,t) + \frac{g}{6} s(r,t)^3 \right] + \zeta(r,t) \tag{2}$$

where the Gaussian random field $\psi(r)$ models the effect of impurities and $\zeta(r,t)$ denotes a Gaussian random force. The random variables ψ and ζ have zero mean and the respective correlations

$$\overline{\psi(r)\psi(r')} = f\delta(r-r') \tag{3}$$

and

$$\langle \zeta(r,t)\zeta(r',t')\rangle = 2\lambda\delta(r-r')\delta(t-t'). \tag{4}$$

Static properties of the model can be studied by means of the n-replica Hamiltonian [6]

$$\mathcal{H}_n[s] = \int \mathrm{d}^d r \left[\sum_{\alpha=1}^{n} \left(\frac{\tau}{2} s_\alpha^2 + \frac{1}{2}(\nabla s_\alpha)^2 + \frac{g}{4!} s_\alpha^4 \right) - \frac{f}{8}\left(\sum_{\alpha=1}^{n} s_\alpha^2 \right)^2 \right] \tag{5}$$

where the components $\{s_\alpha\}_{\alpha=1\ldots n}$ correspond to different replicas of the system. At the end of the calculation one takes the limit $n \to 0$.

The renormalized field $s_R = Z_s^{-1/2} s$, the coupling constants f_R and g_R, and the mass m are defined by the renormalization prescriptions

$$\begin{aligned}
&\Gamma_2^R(p=0; m, f_R, g_R) = m^2 \\
&\frac{\mathrm{d}}{\mathrm{d}p^2}\Gamma_2^R(p; m, f_R, g_R)\bigg|_{p=0} = 1 \\
&\Gamma_{4,g}^R(\{p=0\}; m, f_R, g_R) = g_R m^\epsilon \\
&\Gamma_{4,f}^R(\{p=0\}; m, f_R, g_R) = -3 f_R m^\epsilon
\end{aligned} \tag{6}$$

where Γ_n^R is a renormalized one-particle irreducible n-point vertex function calculated with the Hamiltonian (5). The function $\Gamma_{4,f}^R$ denotes the O(n)-symmetric part of the four-point vertex function while $\Gamma_{4,g}^R$ is the anisotropic part. The renormalization scheme (6) was used by Mayer [7] to calculate the static critical exponents of the dilute Ising model in a four-loop approximation.

To study the critical dynamics it is convenient to cast the Langevin equation (2) in the form of the dynamic functional [8–11]

$$\mathcal{J}[\tilde{s}, s] = \int \mathrm{d}^d r \left[\int \mathrm{d}t\, \tilde{s} \left(\partial_t s + \lambda(\tau - \Delta)s + \frac{\lambda g}{6} s^3 - \lambda \tilde{s} \right) - \frac{\lambda^2 f}{2} \left(\int \mathrm{d}t\, \tilde{s} s \right)^2 \right] \tag{7}$$

where we have already performed the average with respect to disorder [12]. The response field $\tilde{s}$ has been introduced to average over the thermal noise.

The computation of the dynamic critical exponent requires the introduction of a renormalized response field $\tilde{s}_R = Z_{\tilde{s}}^{-1/2}\tilde{s}$ and a renormalized Onsager coefficient $\lambda_R = (Z_{\tilde{s}}/Z_s)^{1/2}\lambda$. The parameter $Z_{\tilde{s}}$ will be fixed by the condition

$$\frac{\mathrm{d}}{\mathrm{d}(\mathrm{i}\omega)}[G_{1,1}^R(p=0,\omega; m, f_R, g_R)]^{-1}\bigg|_{\omega=0} = 1 \tag{8}$$

where $G_{1,1}^R$ is the Fourier transform of the response function $\hat{G}_{1,1}^R(r,t) = \langle s_R(r,t)\tilde{s}_R(0,0)\rangle$.

To simplify the final expressions it is convenient to absorb a geometrical factor into the coupling coefficients:

$$u = 3G_\epsilon g_R \qquad v = 8G_\epsilon f_R \qquad G_\epsilon = (4\pi)^{-d/2}\Gamma((6-d)/2). \tag{9}$$

The Callan–Symanzik functions

$$\beta_u(u,v) = m\frac{\mathrm{d}}{\mathrm{d}m}\bigg|_{f,g} u \qquad \beta_v(u,v) = m\frac{\mathrm{d}}{\mathrm{d}m}\bigg|_{f,g} v \tag{10}$$

and the Wilson function

$$\eta(u, v) = m\frac{\mathrm{d}}{\mathrm{d}m}\bigg|_{f,g} \ln Z_s \qquad (11)$$

have been calculated up to four-loop order in [7]. The dynamic critical exponent is the value of the function

$$z(u, v) = 2 + (\tilde{\eta}(u, v) - \eta(u, v))/2 \qquad \text{with} \qquad \tilde{\eta}(u, v) = m\frac{\mathrm{d}}{\mathrm{d}m}\bigg|_{f,g} \ln Z_{\tilde{s}} \qquad (12)$$

at the fixed point $(u_\star, v_\star)$ defined by $\beta_u(u_\star, v_\star) = \beta_v(u_\star, v_\star) = 0$.

Condition (8) leads to

$$\tilde{\eta}(u, v) + \eta(u, v) = \tfrac{1}{2}v + \tfrac{3}{16}f(d)v^2 - \tfrac{1}{6}f(d)uv + \tfrac{2}{9}\psi(d)u^2 + \tfrac{1}{54}\phi(d)u^3 + \mathrm{O}(v^3, v^2u, uv^2, u^4). \qquad (13)$$

With the Wilson function

$$\eta(u, v) = (\tfrac{1}{54}u^2 - \tfrac{1}{24}uv + \tfrac{1}{64}v^2)h(d) + (\tfrac{1}{216}u^3 - \tfrac{1}{64}u^2v + \tfrac{1}{64}uv^2 - \tfrac{1}{256}v^3)j(d) + \mathrm{O}(\text{four-loop}) \qquad (14)$$

we obtain

$$z(u, v) = 2 + \tfrac{1}{4}v + \tfrac{1}{64}(6f(d) - h(d))v^2 - \tfrac{1}{24}(2f(d) - h(d))uv + \tfrac{1}{54}(6\psi(d) - h(d))u^2 + \tfrac{1}{216}(2\phi(d) - j(d))u^3 + \mathrm{O}(v^3, v^2u, uv^2, u^4). \qquad (15)$$

The main difference between equation (15) and the result obtained by PV is (the absence of) an infrared pole at $d = 2$. The values of the functions $f(d)$, $h(d)$, etc for $d = 2, 3, 4$ are given in table 1. The two-loop integrals are defined by the equations

$$f(d) = \frac{4-d}{G_\epsilon^2}\int_{p,k}\frac{1}{(1+k^2)^2(1+p^2)[1+(p+k)^2]} - \frac{2}{4-d} \qquad (16)$$

$$h(d) = \frac{2(4-d)}{G_\epsilon^2}\int_{p,k}\left(\frac{4}{d}\frac{1}{1+p^2} - \frac{4-d}{d}\right)\frac{1}{(1+k^2)(1+p^2)^2[1+(p+k)^2]} \qquad (17)$$

$$\psi(d) = \frac{4-d}{G_\epsilon^2}\int_{p,k}\frac{1}{(1+p^2)(1+k^2)[3+p^2+k^2+(p+k)^2]^2}. \qquad (18)$$

The three-loop functions $j(d)$ and $\phi(d)$ are more complicated expressions which are not shown here. In the minimal renormalization scheme [13] they are replaced by $j_{\mathrm{MR}} = j(4) - 4h'(4) = -1$ and $\phi_{\mathrm{MR}} = \phi(4) - 12\psi'(4) = -1.1367\ldots$, respectively.

Table 1. The values of the functions occurring in $z(u, v)$ (equation (15)) for $d = 2$, 3 and 4. If the minimal renormalization scheme (MR) is applied the functions have to be replaced by the numbers shown in the last row.

d	$f(d)$	$h(d)$	$j(d)$	$\psi(d)$	$\phi(d)$
4	1	1	2.250 42	ln(4/3)	1.576 68
3	2/3	16/27	0.197 472	0.174 358	0.283 714
2	0.562 605	0.458 543	−0.054 609 0	0.137 292	0.029 944 5
MR	1	1	−1	ln(4/3)	−1.136 76

To determine the dynamic exponent one has to insert the fixed point into equation (15). For the dilute Ising model the fixed point has been computed by a four-loop expansion in conjunction with a Padé–Borel approximation [7] with the result $u_\star = 2.236\,11$, $v_\star = 0.683\,88$. This yields for the critical exponent in three dimensions:

$$z(d=3) = \begin{cases} 2.171 & \text{(one-loop)} \\ 2.191 & \text{(two-loop).} \end{cases} \tag{19}$$

The difference between the values at one- and two-loop order may be used as a rough estimate of the accuracy of the result.

Inserting the fixed points $u_\mathrm{I}(d=3) = 1.429\,93$ and $u_\mathrm{I}(d=2) = 3.767\,04$ [7] of the pure Ising model into $z(u,0)$ we get

$$z_\mathrm{I}(d=3) = \begin{cases} 2.017 & \text{(two-loop)} \\ 2.022 & \text{(three-loop)} \end{cases} \tag{20}$$

and the two-dimensional estimate

$$z_\mathrm{I}(d=2) = \begin{cases} 2.096 & \text{(two-loop)} \\ 2.124 & \text{(three-loop)} \end{cases} \tag{21}$$

respectively.

Due to a slow crossover observed in dilute Ising systems [14, 3] it is difficult to determine asymptotic critical exponents by Monte Carlo methods. Heuer [14] finds $z = 2.4 \pm 0.1$ for the three-dimensional system, while a Monte Carlo renormalization group approach [15] gave $z = 2.20 \pm 0.07$ (at low dilutions). Only the latter result is consistent with the estimates $z \simeq 2.19$ obtained above and $z \simeq 2.18$ found in [3].

For the pure Ising model the three-loop approximations (20) and (21) are close to the estimates $z_\mathrm{I}(d=3) = 2.019$ and $z_\mathrm{I}(d=2) = 2.126$ of [1]. Li *et al* [16] have measured the dynamic exponent for $d = 2$ in the initial stage of a relaxation process and found the result $z_\mathrm{I}(d=2) = 2.1337 \pm 0.0041$. A recent computer simulation [17] using the concept of 'damage spreading' gave $z_\mathrm{I}(d=3) = 2.032 \pm 0.004$ and $z_\mathrm{I}(d=2) = 2.172 \pm 0.006$. A review of the Monte Carlo methods which have been employed to find estimates for z_I is also given in [18] and [19].

It is a pleasure to thank H K Janssen for interesting discussions and St Theiss for a critical reading of the manuscript. This work has been supported by Sonderforschungsbereich 237 (Unordnung und große Fluktuationen) of the Deutsche Forschungsgemeinschaft.

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